On the divergence calculation of the Extended Kalman Filtering for chaotic signals

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***Abstract*—The Extended Kalman Filter (EKF) has become one of the options to achieved synchronization of chaotic signals with important applications for wireless communications. However the joint operation of the EKF and the chaotic signals suffers from several reported drawbacks that negatively affect the performance of the EKF up to a breaking point. This paper shows a rather simple way of establishing a reference threshold that allows to avoid the breaking point and keeps the EKF in a working regime that yields good performance in terms of the Mean Square Error (MSE) between the real and estimated chaotic signals. Three different chaotic systems are used to illustrate the proposed approach.**

Keywords: Extended Kalman Filter; Strange Attractors; Lorenz Model; Rossler Model; Chua Model; Synchronization; Chaotic Communications.

I. INTRODUCTION

A chaotic system is a non-linear model with strong dependence and sensitivity on its initial conditions a feature that, for some time, seemed to defy synchronization between two chaotic systems. The discovery of the self-synchronization property of some chaotic systems [1] is acknowledge as the "entrance pass" of chaos in the communications arena [2] with applications that go from modulation [3, 4] to secure communication [5, 6] just to mention some of many more works in this line.

The synchronization can be achieved in different ways. The first proposed synchronization approach [1] assumes a decomposition of the chaotic system into a driving system (which can be regarded as the transmitter) and a stable state response system (say the receiver) that get synchronized by means of a common signal. Synchronization can also be achieved by means of the EKF [2, 3, 7-9] which is close to the purpose of this work.

The EKF has two main drawbacks reported in the literature that reduce the performance. On one side when the EKF is applied for chaotic systems, the Kalman gain either oscillates or converges to a fixed point greater than zero [10]. On the other side when the EKF works in presence of non-linear systems it tends to become too confident on its estimates, which prevents a correct update based on new data and provokes the filter to lock into a parameter estimate with a large error leading to ill-conditioned error covariance matrices. This phenomenon is known as divergence [11, 12]. Indeed the divergence phenomenon is inherited from the standard Kalman Filter where the problem is somehow compensated by the square root filtering using the Cholesky factorization [13]. The same Cholesky factorization is applied in the EKF but it is not the absolute and ultimate solution and eventually the error covariance matrix becomes non-positive definite crashing any numerical calculation. In chaotic communication systems this problem is of paramount importance. For the case of the EKF working with chaotic signals this work presents a way to "forecast" the time when the ill-condition of the error covariance matrix may take place. The calculation involves the Lyapunov exponents of the chaotic model.

II. PROPOSED APPROACH

We applied the time update and measurement update equations of the EKF [7] for 3 chaotic models with the following parameters [14]. For Rössler a = 0.15, b = 0.2, c=10. For Lorenz σ = 16, R = 45.92, b = 4 and Chua α = 9.2, β= 14.3, a = -1/7, b = 2/7.

The approach is based on two main considerations. First, in information theoretic terms the magnitudes of the Lyapunov exponents (given in bits/s) quantify the dynamics of the chaotic system [14]. This means that the future behavior can be "*forecasted*" only up to a certain time instant which depends on the value of the positive Lyapunov exponent (which indicates exponential expansion) and the accuracy used to establish the initial conditions [14]. Second, the divergence of the error covariance matrix in the Kalman Filter based approaches is due to finite word length numerical computations [13].

The first consideration refers to the fact that the accuracy used to specify the initial conditions between two (parametrically identical) chaotic systems determine the point when the divergence between the two of them will take place (sensitivity to initial conditions).

Let us make the following interpretation. Let it be a source chaotic system. The EKF generates its own version of the chaotic system not necessarily with identical parameters but, for the moment suppose that parameters for both chaotic systems are the same. The EKF is able to synchronize both chaotic systems in very short time no matter if the initial conditions are completely different or rather close to each other but, the two chaotic processes will diverge after some time. So what relationship, if any, exists between this and the time when the error covariance matrix becomes ill-posed?

Let us investigate if the time when the covariance matrix becomes ill-posed (divergence time, D_T) can be evaluated according to:

$$D_T = \frac{\Delta I_C}{\lambda_1} \qquad (1)$$

where ΔI_C is the accuracy for the initial conditions, and λ_1 is the value of the largest positive Lyapunov exponent expressed in bits/s.

Let us assume that numerical evaluations are made using MatLab with double precision floating point representation of numbers. In this case the numerical data is build according to the IEEE-754 standard for double precision which requires 64 bits distributed as follows: 1 bit for the sign, 11 bits for the exponent and 52 bits for the Mantissa. In this case accuracy for the initial conditions will be $\Delta I_C = 52$ bits. Next, for the Rössler, Lorenz and Chua strange attractors, the following table shows the value of the largest Lyapunov exponent [14].

	λ_1	λ_1 (bits/s)
Rössler	0.09	0.13
Lorenz	1.5	2.16
Chua	0.326	0.47

Table1. Positive Lyapunov exponents.

Upon substitution of (1) one gets the approximated divergence time of the EKF under the presence of signals from Rössler, Lorenz and Chua strange attractors:

	D_T
Rössler	400 sec
Lorenz	24 sec
Chua	110 sec

Table 2. Theoretical divergence time D_T.

According to our simulation results, for the case of the 3 chaotic attractors under study, the Kalman gain of the EKF shows the behavior reported in [10] where the Kalman gain converges to a certain fixed value (shown in the full version).

After some running time the EKF yields a non positive definite error covariance matrix for all the cases under study crashing the simulation. Close examination of this situation revealed that for a given component of the attractor (x, y or z) the correspondent value in the error covariance matrix starts to be negative at a given point and from then on it shows unstable variations for some time up to the point when it assumes a negative value bigger than the variance of the measurement noise. At this point the error covariance matrix is ill-posed making impossible to make the Cholesky factorization.

For the three cases under study the instant when certain element in the error covariance matrix (let us say for the x component) becomes negative for the first time was recorded. Let us denote it as the experimental divergence time D_{TE}. We denote the Signal to Noise Ratio (S/N) as the ratio of the average power of the chaotic signal (say for the x component) and the measurement noise. For three different values of the S/N, with the correspondent sampling times T_S, the table 3 shows the D_{TE} which closely follows the calculated D_T.

S/N	0.1	1	10
Rössler $T_S = 10^{-2}$ sec	340 sec	330 sec	300 sec
Lorenz $T_S = 10^{-3}$ sec	23.9 sec	22.95 sec	21.6 sec
Chua $T_S = 10^{-2}$ sec	99 sec	93 sec	91 sec

Table3. Experimental divergence time D_{TE}.

It is worth mentioning that the evaluation of the divergence time based on the Lyapunov exponents has to be considered as an upper boundary.

III. Contents of the full version

The full version of the paper will show that for practical cases it is possible to set a divergence prediction threshold 25% percent below of the calculated D_T. In this moment the fatal journey of the error covariance matrix can be stopped by restoring the conditions when the EKF began its operation, concretely a unitary error covariance matrix. This mid-flight "*resynchronization*" does not affect significantly the performance of the EKF. The MSE between the original signals and the estimated signals will be presented, showing that the MSE without the "*readjustment*" (before the error covariance matrix gets its first negative element) and the MSE with continuous "*resynchronized*" regime are practically the same.

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